On the cobordisms of Möbius circles

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Abstract

The boundary of a Möbius manifold carries a canonical Möbius structure. This enables one to define the cobordism group of n-dimensional (closed) Möbius manifolds. The purpose of this note is to show that the cobordism group of Möbius circles is zero, i.e., every Möbius circle bounds a Möbius surface. We also complete N. Kuiper's classification of projective structures on S^1 (we show that there are in fact two series of projective circles with parabolic holonomy, and not one).

1 Introduction

Recall that a *Möbius* manifold is an oriented (G, X)-manifold (see e.g., [3, 3.1.2]) of dimension n with $X = S^n$ and G the group $\mathrm{M\ddot{o}b}_n$ of orientation-preserving Möbius transformations (see e.g., [1] for details). We suppose that the charts participating in the definition of the Möbius structure are orientation preserving. Two Möbius structures on the same manifold are said to be *equivalent* if one of them is induced from the other under some diffeomorphism homotopic to the identity.

For a Möbius manifold M, we set $-M$ to be the Möbius manifold "inverse to M", i.e., as a smooth oriented manifold, $-M$ is equal to M with the orientation reversed, and the charts that define the Möbius structure on $-M$ are the compositions of the charts for M and an orientation-reversing Möbius transformation. In the sequel, all Möbius diffeomorphisms are assumed to be orientation preserving. A diffeomorphism $M_1 \to M_2$ between two Möbius manifolds is said to be *anti-Möbius*, if the composition $-M_1 \xrightarrow{\mathrm{Id}} M_1 \to M_2$ is Möbius.

The definition of Möbius structures can be naturally extended to manifolds with boundary. Let M is a Möbius manifold, and let $p : N_1 \to N_2$ is an anti-Möbius diffeomorphism between two disjoint components of ∂M. Then the manifold M' obtained from M by identifying x and $p(x)$ for all $x \in N_1$ can be equipped with a canonical Möbius structure.

Hence, one can define the *cobordism group* $\Omega_n^{\mathrm{M\ddot{o}b}}$ *of n-dimensional Möbius manifolds* as follows. We set the elements of $\Omega_n^{\mathrm{M\ddot{o}b}}$ to be n-dimensional closed Möbius manifolds considered modulo the equivalence relation $\sim$ with $M_1 \sim M_2$, iff there exists an $n + 1$-dimensional compact Möbius manifold N such that ∂N is Möbius diffeomorphic to $M_1 \sqcup -M_2$. The group operation is induced by taking the disjoint union. The main result of this note is the following theorem.

Theorem 1 $\Omega_1^{\mathrm{M\ddot{o}b}} = 0$.

The proof is straightforward: we describe in section 2 the Möbius structures on the circle, and for each of those we construct in section 3 a bounding surface. In section 2, we also show (lemma 1 and remark 2) that some cases are missing from N. Kuiper's classification [2] of projective structures on the circle, and we complete that classification.

Remark 1. One can define for any $n \geq 1$ the cobordism category $\mathcal{MC}_n$ of n-dimensional Möbius manifolds as follows. We set the objects of $\mathcal{MC}_n$ to be compact Möbius manifolds of dimension n, and for any two objects N_1 and N_2 of $\mathcal{MC}_n$, morphisms from N_1 to N_2 are defined to be $n+1$-dimensional compact Möbius manifolds N such that $\partial N = N_1 \sqcup -N_2$ (modulo Möbius diffeomorphisms that are identical on ∂N). The categories $\mathcal{MC}_n$ are strict symmetric monoidal, and the above theorem 1 can be reformulated as $K_0(\mathcal{MC}_1) = 0$. It would be interesting to learn more about $\mathcal{MC}_1$, e.g., a presentation by generators and relations etc.

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2 Möbius structures on the oriented circle

In this section we recall the classification of Möbius structures on the circle S^1. A Möbius structure on a 1-dimensional manifold can be viewed as a (G, X)-structure, where X is any circle C in $\mathbb{C}P^1$, and G is the group of automorphisms of $\mathbb{C}P^1$ that preserve both disks bounded by C. We shall often use the projective model, i.e., we shall take the circle $\mathbb{R}P^1 \subset \mathbb{C}P^1$, so that G becomes the group of orientation-preserving projective automorphisms of $\mathbb{R}P^1$. Given a Möbius structure S on S^1, we denote by $[S]$ the equivalence class of S, and we define $\mathcal{M}(S^1)$ to be the set of all equivalence classes of Möbius structures on S^1.

Let us identify the universal cover of S^1 with the (canonically oriented) $\mathbb{R}$. Any Möbius structure S on S^1 constructed below will be defined by an orientation-preserving developing map F_S from $\mathbb{R}$ to a model circle $C \subset \mathbb{C}P^1$ and a holonomy homomorphism f_S from $\mathbb{Z}$ to the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ corresponding to C; the maps F_S and f_S should satisfy the condition

$$F_S(x+1) = f_S(1) \cdot F_S(x). \tag{1}$$

Let A_S be a matrix that represents $f_S(1)$ (this matrix is defined up to a sign). We shall distinguish the following cases:

1. $|tr(A_S)| \leq 2$, and A_S is diagonalisable.

2. $|tr(A_S)| = 2$, A_S is nondiagonalisable.

3. $|tr(A_S)| > 2$.

The structures that verify (1), (2), (3), will be called respectively *elliptic, parabolic* and *hyperbolic*.

For any $\alpha > 0$ and $n > 0$ let $\psi_{n,\alpha}$ be a continuous map $[0,1] \to \mathbb{R}P^1$ such that 1. $\psi_{n,\alpha} = e^{\alpha x} - 1$, when x is close to 0, $\psi_{n,\alpha} = e^{\alpha}(e^{\alpha x} - 1)$, when x is close to 1, 2. $\psi_{n,\alpha}^{-1}(0)$ consists of $n+1$ elements, and 3. $\psi_{n,\alpha}|(0,1)$ is an orientation-preserving local diffeomorphism. Analogously, for any couple (n, ε) (where $n > 0$ is an integer, and $\varepsilon = \pm 1$) let $\xi_{n,\varepsilon}$ be a continuous map $[0,1] \to \mathbb{R}P^1$ such that 1. $\xi_{n,\varepsilon} = -\frac{1}{x}$, when x is close to 0, $\xi_{n,\varepsilon} = \varepsilon + \frac{1}{1-x}$, when x is close to 1, 2. $\xi_{n,\varepsilon}^{-1}(\infty)$ consists of $n+1$ elements, and 3. $\xi_{n,\varepsilon}|(0,1)$ is an orientation-preserving local diffeomorphism. The following theorem gives an explicit representative for each element of $\mathcal{M}(S^1)$ (cf. [2]).

Theorem 2 *1. If two Möbius structures on S^1 belong to different types, they are not equivalent.*

2. *Equivalence classes of hyperbolic Möbius structures are parametrised by couples (n, α), where α is a positive real number, and n is a nonnegative integer.*

   The corresponding maps $F_S : \mathbb{R} \to \mathbb{R}P^1, f_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ can be chosen as follows. Let $f_S(1)$ be the map $z \mapsto \mathrm{e}^\alpha z$. If $n = 0$, set $F_S(x) = \mathrm{e}^{\alpha x}$, otherwise set $F_S(x) = \psi_{n,\alpha}(x)$ for $x \in [0,1]$ and extend F_S to $\mathbb{R}$ using (1). Denote the equivalence class of hyperbolic Möbius structures corresponding to the couple (n, α) by $H_{n,\alpha}$.

3. *Equivalence classes of parabolic Möbius structures are parametrised by elements of the set $(\{nonnegative\ integers\} \times \{\pm 1\}) \setminus \{(0, -1)\}$.*

   The equivalence class that corresponds to the couple $(0, 1)$ is represented by the Möbius structure defined by the following maps $F_S : \mathbb{R} \to \mathbb{R}P^1, f_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$: set $F_S(x) = x, f_S(1)(z) = z + 1$.

   The equivalence class corresponding to the couple (n, ε) (where n is a positive integer, and ε is 1 or -1) is represented by the Möbius structure defined by the maps $F_S : \mathbb{R} \to \mathbb{R}P^1, F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ that can be constructed as follows: set $f_S(1)(z) = z + \varepsilon$, set $F_S(x) = \xi_{n,\varepsilon}(x)$ for $x \in [0,1]$ and extend F_S to $\mathbb{R}$ using (1). Denote the equivalence class of parabolic Möbius structures corresponding to the couple (n, ε) by $P_{n,\varepsilon}$.

4. *Equivalence classes of elliptic Möbius structures are parametrised by positive real numbers. The equivalence class corresponding to $\alpha > 0$ can be represented by the Möbius structure defined by the maps $F_S : \mathbb{R} \to \{complex\ numbers\ of\ modulus\ 1\}$ and $f_S(1) : \mathbb{Z} \to \mathrm{PU}_{1,1}$ given by $F_S(x) = \mathrm{e}^{i\alpha x}, f_S(1)(z) = \mathrm{e}^{i\alpha} z$. Denote this equivalence class by E_α.*

Proof of theorem 2. The argument is analogous to the one used in [2] to classify projective circles modulo projective diffeomorphisms (but see remark 2 below).

First, let us show that any Möbius structure S on S^1 is equivalent to some structure described in theorem 2. Suppose, e.g., that S is hyperbolic. Choose a base point in S^1, and denote by F_S the corresponding developing map $\mathbb{R} \to \mathbb{R}P^1$; let $f_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ be the homomorphism such that F_S and f_S satisfy (1). Then, replacing S by an equivalent structure, we can assume that f_S is given by the formula $f_S(n)(z) = \mathrm{e}^{\alpha n} z$ for some $\alpha > 0$. The image of F_S is either the whole $\mathbb{R}P^1$ or not; we can assume (changing the base point, if necessary) that $F_S(0)$ is 1 in the first case and 0 in the second. Now it is easy to see that S is equivalent to some structure from the second assertion of theorem 2. The parabolic and elliptic cases are considered in an analogous way.

Now, let us prove that the structures introduced in theorem 2 are pairwise nonequivalent. Let S be a Möbius structure on S^1. Suppose that S is defined by an orientation-preserving local diffeomorphism $F_S : \mathbb{R} \to \mathbb{R}P^1$ and a homomorphism $f_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ that satisfy (1). We can associate the following invariants to S: the conjugacy class of $f_S(1)$ in $\mathrm{PSL}_2(\mathbb{R})$ and the number

$$\max_{x \in \mathbb{R}, y \in \mathbb{R}P^1} \#(F_S^{-1}(y) \cap [x, x + 1]).$$

It can be easily checked that these invariants are sufficient to distinguish any two different structures defined in theorem 2.$\diamond$

A *projective* structure on the circle is an $(\mathrm{PGL}_2(\mathbb{R}), \mathbb{R}P^1)$-structure. Every Möbius circle is a projective circle; conversely, one can orient a projective circle so that it will become a Möbius circle.
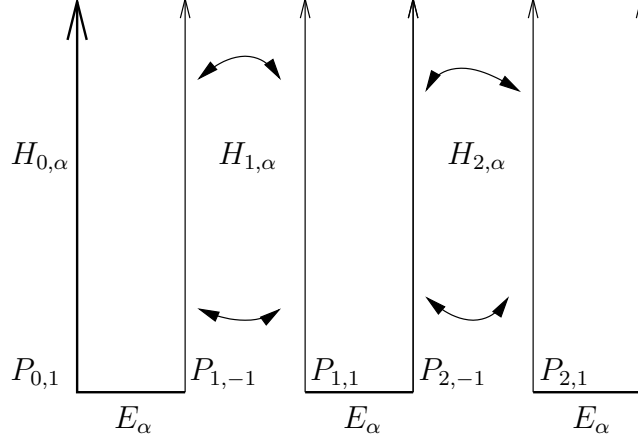
Figure 1: $\mathcal{M}'(S^1)$

Lemma 1 *Any two different Möbius structures on S^1 defined in theorem 2 are not equivalent as projective structures.*

Proof of Lemma 1. We proceed as in the proof of theorem 2. Suppose that the projective structure S on S^1 is defined by a couple (F_S, f_S), where F_S is a local diffeomorphism $\mathbb{R} \to \mathbb{R}P^1$ and $f_S : \mathbb{Z} \mapsto \mathrm{PSL}_2(\mathbb{R})$ is a homomorphism that satisfy (1). It can be easily checked that the conjugacy class of $F_S(1)$ in $\mathrm{PGL}_2(\mathbb{R})$ and the number

$$\max_{x \in \mathbb{R}, y \in \mathbb{R}P^1} \#(F_S^{-1}(y) \cap [x, x+2])$$

depend only on the equivalence class of S. These invariants of projective structures distinguish any two nonequivalent Möbius structures on S^1.$\diamondsuit$

Remark 2. Due to Lemma 1 there exists a bijection $\mathcal{M}(S^1) \leftrightarrow$ (projective circles modulo projective diffeomorphisms). Hence, a classification of projective circles can be obtained from theorem 2; the classification given in [2] in not quite correct: if $n > 0$, then S^1 provided with a Möbius structure of class $P_{n,1}$ and S^1 provided with a Möbius structure of class $P_{n,-1}$ are not isomorphic as projective circles.

Remark 3. The interpretation in terms of developing maps allows us to introduce a topology on $\mathcal{M}(S^1)$. However, the resulting topological space is nasty. The space $\mathcal{M}'(S^1) = \mathcal{M}(S^1) \setminus \{E_{2\pi k} | k$ is an integer $> 0\}$ is the non-Hausdorff topological 1-manifold obtained by identifying the thin lines on Figure 1; $\mathcal{M}'(S^1)$ is open in $\mathcal{M}(S^1)$, and for any $k > 0$ we can take the system of sets of the form $\{E_{2\pi k}\} \cup$ (a neighbourhood of $P_{k,1}$ in $\mathcal{M}'(S^1)) \cup$ (a neighbourhood of $P_{k,-1}$ in $\mathcal{M}'(S^1)$) as a local neighbourhood basis at $E_{2\pi k}$.

3 Proof of theorem 1.

In order to prove theorem 1, it is enough to show that any element of $\mathcal{M}(S^1)$ is equal to $[S]$, where S is the Möbius structure on the boundary of some 2-dimensional Möbius surface. Notice that a Möbius structure on a 2-dimensional manifold is the same as a $(\mathrm{PSL}_2(\mathbb{C}), \mathbb{C}P^1)$-structure.

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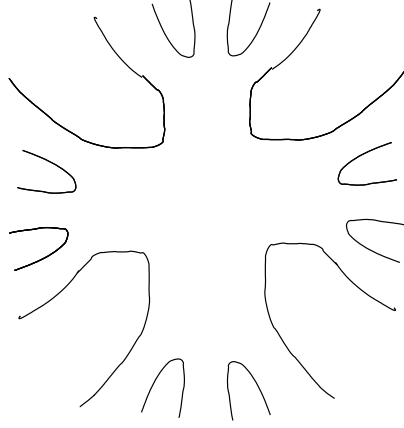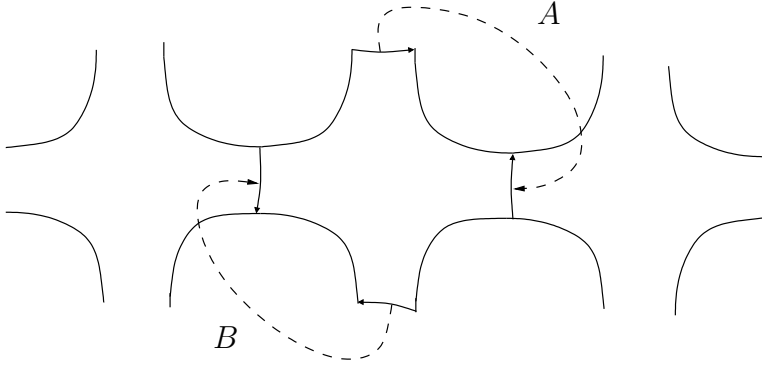
Figure 2:

Figure 3:

Let N be the sphere minus three disjoint open disks ("the pants"). The manifold $\tilde{N}$ is represented on Figure 2.

Denote by $\mathbf{F}(A, B)$ the free group on the generators A, B. Suppose that $\mathbf{F}(A, B)$ acts on $\tilde{N}$ as shown on Figure 3 (the octagon in the middle is a fundamental domain; the right and the left octagons are the images of the fundamental domain under A and B respectively).

We shall equip N (and hence, ∂N) with a Möbius structure by constructing maps $F : \tilde{N} \to \mathbb{C}P^1, f : \mathbf{F}(A, B) \to \mathrm{PSL}_2(\mathbb{C})$ that satisfy

$$F(g \cdot x) = f(g) \cdot x, g \in \mathbf{F}(A, B), x \in \tilde{N}.$$

This can be done by choosing two elements $a, b \in \mathrm{PSL}_2(\mathbb{C})$ and an embedded rectangular octagon in $\mathbb{C}$ with vertices $X_1, X_2, \ldots, X_8$ (see Figure 4; arrows indicate the orientations of the boundary) such that the following conditions are satisfied:

(C1) All segments $X_1 X_2, \ldots, X_8 X_1$ are arcs of circles; denote the corresponding circles by $C_{X_1 X_2}, \ldots, C_{X_8 X_1}$.

Figure 4:

(C2) $a(X_1) = X_4, a(X_2) = X_3, b(X_5) = X_8, b(X_6) = X_7$.

(C3) $a(C_{X_2 X_3}) = C_{X_2 X_3}, a(C_{X_8 X_1}) = C_{X_4 X_5}, b(C_{X_6 X_7}) = C_{X_6 X_7}, b(C_{X_4 X_5}) = C_{X_8 X_1}$.

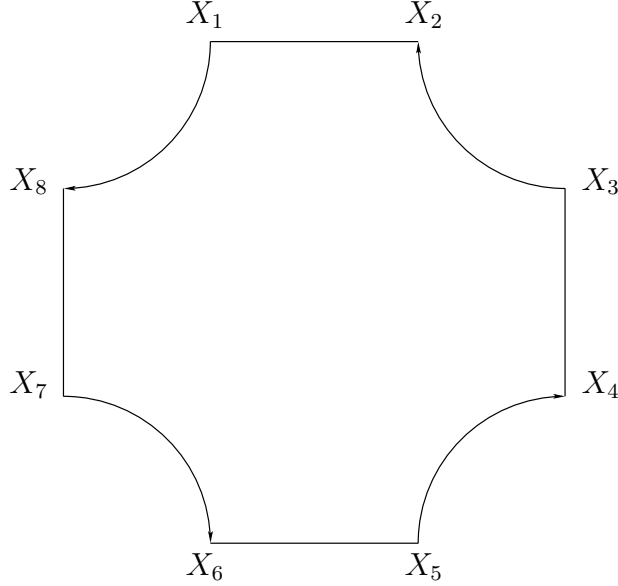Let x be a positive real number. Define the elements $a_x, b \in \mathrm{PSL}_2(\mathbb{C})$ respectively by $z \mapsto z + x, z \mapsto \frac{z}{1-z}$. The transformation $b \circ a_x$ is represented by a matrix with trace $2 - x$. Choose the octagon as shown in Figure 5. The horizontal line at the bottom is the real axis, the circles C_1 and C_2 on Figure 5 are defined respectively by the equations $(\mathrm{Re}\, z)^2 - 2\,\mathrm{Re}\, z + (\mathrm{Im}\, z)^2 = 0$ and $(\mathrm{Re}\, z)^2 + 2\,\mathrm{Re}\, z + (\mathrm{Im}\, z)^2 = 0$. The circles C_1' and C_2' are chosen in such a way that 1. $b(C_1') = C_2', a_x(C_2') = C_1'$, 2. C_1' is orthogonal to C_1 and depends continuously on x, 3. $C_1' \subset \{z \mid \mathrm{Re}\, z \geq 0\}$, and the imaginary part of the center of C_1' is > 0. Note that there are many ways to choose the circles C_1' and C_2'' that satisfy these conditions. The arc $X_6 X_7$ is an arc of a sufficiently small circle in the upper half-plane that is tangent to the real axis at 0.

Note that we have $b(z) = -\bar{z}$ for $z = 1 + \mathrm{e}^{it}$, i.e., the restriction of b to C_1 is the symmetry with respect to the imaginary axis. This implies easily that for any $x > 0$, the octagon on Figure 5 and the maps $a = a_x$ and b satisfy the conditions (C1)-(C3).

For any $x > 0$ we obtain Möbius structures on N and all components of ∂N. The components of ∂N that correspond to $X_2 X_3$ and $X_6 X_7$ will always have parabolic structures that belong to $P_{0,1}$, and the equivalence class of the structure on the third component changes as we change x. Denote this structure by S_x; S_x is elliptic for $x < 4$, parabolic for $x = 4$ and hyperbolic for $x > 4$. Denote by $N_1(x)$ the Möbius surface obtained by gluing together two parabolic components of ∂N (note that due to Lemma 1, for any Möbius circle C there exists an anti-Möbius diffeomorphism $C \to C$).

Let us determine the equivalence class of S_x. Let F_{S_x} be a developing map $\mathbb{R} \to \mathbb{R}P^1$ that corresponds to S_x.

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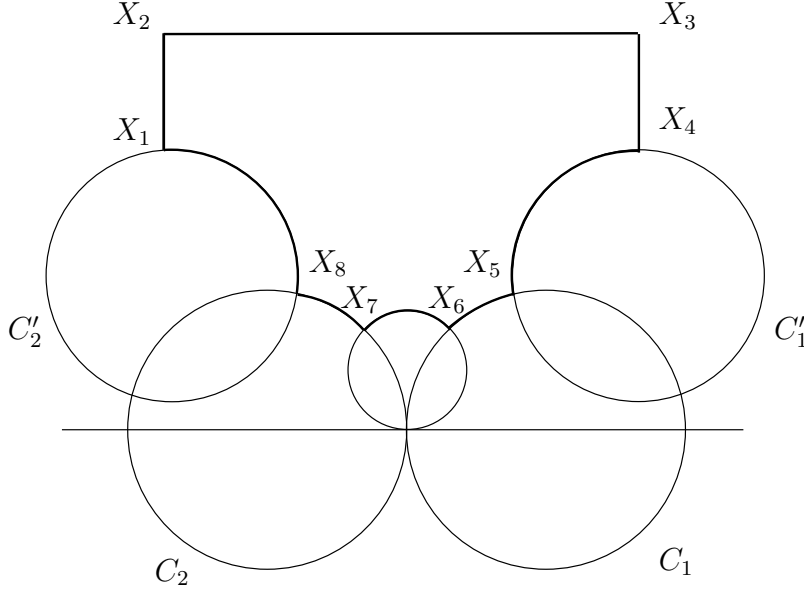
Figure 5:

Note that $[S_4] = P_{0,1}$, which implies that the image of F_{S_4} is not the whole $\mathbb{R}P^1$. Hence, if $D > 0$, then we have $F_{S_x}([0, D]) \neq \mathbb{R}P^1$ for any x sufficiently close to 4. This implies the following lemma:

Lemma 2 *For any sufficiently small $\alpha > 0$ there exist x_1, x_2 such that the Möbius structure on $\partial N_1(x_1)$ (respectively, $\partial N_1(x_2)$) belongs to $H_{0,\alpha}$ (respectively, to E_α).*

Let us define the action of $\mathbb{N}$ on $\mathcal{M}(S^1)$ as follows: for any $k \in \mathbb{N}, \alpha > 0, n \geq 0$ and $\varepsilon \in \{\pm 1\}$ set $kH_{n,\alpha} = H_{kn,k\alpha}, kP_{n,\varepsilon} = P_{kn,\varepsilon}, kE_\alpha = E_{k\alpha}$.

Lemma 3 *Let C' and C'' be oriented circles. Suppose that C'' is equipped with a Möbius structure, and denote this structure by S''. Let $p : C' \to C''$ be an orientation-preserving k-sheeted covering. Denote by S' the inverse image of S'' with respect to p. We have $[S'] = k[S'']$.*

$\diamondsuit$

Lemma 4 *Let N be a compact Möbius surface with one boundary component and nontrivial fundamental group, and let $k > 0$ be an integer. Denote by S the Möbius structure on ∂N. There exists a compact Möbius surface N' such that $\partial N'$ is a circle, and the Möbius structure on $\partial N'$ belongs to $k[S]$.*

Proof of Lemma 4. Let $\mathfrak{S}_k$ be the symmetric group on k elements. If k is odd, then there exists a homomorphism $\pi_1(N) \to \mathfrak{S}_k$ that takes a generator of $\pi_1(\partial N)$ to some cycle of length k. Indeed, we can choose a system $v_1, w_1, \ldots, v_g, w_g$ of free generators of $\pi_1(N)$

Figure 6:

so that $\pi_1(\partial N)$ will be spanned by $[v_1, w_1] \cdots [v_g, w_g]$. A required homomorphism can be constructed using the fact that in $\mathfrak{S}_k$ a cycle of length k is a commutator (e.g., we have

$$(1 \ldots k) = \left(1 \frac{k+3}{2} \ldots k\right)\left(12 \ldots \frac{k+1}{2}\right) = \sigma\left(12 \ldots \frac{k+1}{2}\right)^{-1} \sigma^{-1}\left(12 \ldots \frac{k+1}{2}\right)$$

for some $\sigma \in \mathfrak{S}_k$).

Hence, for k odd, there exists a surface N' with one boundary component and a k-sheeted covering $N' \to N$. This, together with Lemmas 2 and 3, allows us to construct for any $\alpha > 0$ a Möbius surface, whose boundary carries a Möbius structure that belongs to E_α.

Now cut out a small Möbius disk from N. The resulting Möbius surface has two boundary components; one of these components has the structure S, and the structure on the other one belongs to $E_{2\pi}$. Note that if M is a compact surface with two boundary components, then a generator of one of these components can be taken as a free generator of $\pi_1(M)$; hence, for any $k > 0$ there exists a k-sheeted covering $M' \to M$ that is cyclic over one of the components of ∂M. We have already constructed for any integer $l > 0$ a Möbius surface, whose boundary carries a structure from $E_{2\pi l}$, and the lemma follows. $\diamondsuit$

Lemmas 2, 3, 4 imply that any of the classes $P_{0,1}, H_{0,\alpha}$ or $E_\alpha, \alpha > 0$ is the equivalence class of the Möbius structure of the boundary of some Möbius surface. In order to complete the proof of theorem 1, we can proceed as follows. Suppose that $x > 2$ is a real number and consider the rectangular octagon represented on Figure 6.

Here the circles C_1 and C_2 are the same as in the proof of Lemma 2. Suppose that the centers of C_1' and C_2' are the points $\pm\frac{x}{2} + i$, and the radii of these circles are $\frac{x}{2} - 1$. As above, the octagon and the maps a_x and b give us for any $x > 2$ a Möbius structure on N (the pants). The induced Möbius structure on the component of ∂N that corresponds to the arc $X_2 X_3$ (respectively, the arc $X_6 X_7$) belongs to the class $P_{0,1}$ (respectively, $P_{1,-1}$). Denote by S_x' the Möbius structure on the third component of ∂N. The structure S_x' is elliptic for $x < 4$; by attaching Möbius surfaces to the elliptic component of ∂N and to the

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$P_{0,1}$-component, we obtain a Möbius surface, whose boundary has a Möbius structure of class $P_{1,-1}$, which means that we can eliminate the $P_{1,-1}$-component as well, i.e., for any $x > 2$ there exists a Möbius surface, whose boundary carries the Möbius structure S'_x.

Let us determine the equivalence class of $S'_x, x \geq 4$. Note that we can choose a developing map $F_{S'_x} : \mathbb{R} \to C'_1$ for the structure S'_x so that $F_{S'_x}$ sends the segment $[0, 1/2]$ to the arc $X_5 X_4$ and the segment $[1/2, 1]$ to the arc $a_x(X_1 X_8)$. It is easy to check that for $x \geq 4$, the fixed points of $a_x \circ b$ are exactly the intersection points of C'_1 and the real axis.

Hence, for any $x > 4$ the set $F_{S'_x}^{-1}$(the fixed points of $a_x \circ b) \cap [0, 1)$ consists of two elements, which implies that for any $x > 4$ we have $S'_x \in H_{1,\alpha}$ with $\alpha = 2 \operatorname{arccosh}(x/2 - 1)$.

An analogous argument shows that $S'_4 \in P_{1,1}$ (note that $a_4 \circ b$ acts on C'_1 "clockwise"). The proof of theorem 1 is easily completed using Lemmas 3 and 4. $\diamondsuit$

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